Theory of Incomplete Models of Dynamic Structures

ALEX BERMAN* AND WILLIAM G. FLANNELLY†
Kaman Aerospace Corporation, Bloomfield, Conn.

A method is presented for identifying parameters in a linear, discrete model of a structure by using measured normal modes to modify an analytically derived model. The structure considered has a relatively large number of points of interest and a frequency range of interest influenced by a relatively small number of normal modes. The analytical model which is introduced has fewer degrees of freedom (normal modes) than coordinates (points of interest). The parameters of this "incomplete model" are obtainable from the limited, but quantitative, test data, and the conceptually valid, but approximate, analytical data. The characteristics of this model and methods of using it are discussed, in particular, means of computing the effects of mass and stiffness changes on natural frequencies and normal modes. Computer experiments illustrate these methods and demonstrate that such a model can be useful and that the procedure is not overly sensitive to measurement errors.

Nomenclature

A = coefficient matrix in mass equation
C = influence coefficient matrix
f = force vector
g = structural damping coefficient
K = stiffness matrix
M = mass matrix
m_i = generalized mass of ith mode
m_A = approximation to $\bar{m}$
$\bar{m}$ = vector consisting of unknown elements of M
N = number of modes
n_e = number of equations
n_v = number of variables
P = number of points of interest (points of measurement)
R = right-hand side of mass equation
W = weighting matrix referring to confidence in m_A (diagonal)
Y = mobility matrix
y = displacement vector
Z = impedance
Δ = any change (as a prefix)
Φ = matrix of modes
ϕ_i = modal vector (ith mode)
Ω_i = natural frequency of ith mode
ω = forcing frequency

Subscript

inc = incomplete model

Superscripts

$+$ = pseudo inverse
$\lceil \ \rfloor$ = indicates diagonal matrix

Introduction

FOR any structure whose dynamic response is an important consideration in its final design, the need for an analytical model is apparent. Although the state-of-the-art of purely analytical modeling is quite advanced, such models do depend to some extent on the analyst's experienced intuition and will not precisely predict the results of actual tests. They do have the capability of predicting the general effects of changes in loading or in the structure itself. Modern test procedures, on the other hand, yield quantitative data about the condition actually tested but can give no direct information regarding other conditions or modified structures.

There have been attempts in recent years to use the results of dynamic testing to identify the parameters in the equations of motion. A survey of the work in this area has been conducted by Young and On.[1] However, no method has been generally accepted as a means of deriving a useful analytical model from test data.

There is a basic and inherent difficulty in attempting to use test data to define a finite degree-of-freedom model of a continuous system. A discrete model has as many natural frequencies as degrees of freedom and these must all be significantly represented in the test data if one is to identify a unique model. Otherwise, there will result a poorly conditioned set of equations whose solution will be extremely sensitive to small measurement errors. In order to obtain such data, the structure must be excited at frequencies encompassing the appropriate number of natural frequencies (whether the excitation is sinusoidal or transient) since only modes in the vicinity of the frequency of excitation will respond significantly. If the number of degrees of freedom is large, the frequency range required can be prohibitively large and well beyond the frequency range of interest.

There are conditions where a unique identification could be possible. One in particular is where the number of points of interest is small and data from the appropriate frequency range is available. Such a situation is treated in Ref. 2.

When the number of spatial coordinates exceeds the number of resonances in the frequency range tested, there are an infinite number of analytical models that will duplicate the measured responses within normal experimental error. It is this presumably-more-common situation that is considered in this paper.

The characteristics required of a model depend to some extent on the use to be made of the structure in question. It is never required, however, that the model behave in every way exactly like the actual structure. There will always be a finite number of "points of interest" on the structure (the points at which measurements would be taken, for example) and there will always be a finite "frequency range of interest" (the frequencies that are expected to be contained in the exciting forces). Any of the infinite number of models that would duplicate the test results discussed above could

Presented at the AIAA/ASME 11th Structures, Structural Dynamics, and Materials Conference, Denver, Colo., April 22-24, 1970 (no paper number; published in bound volume); submitted May 14, 1970; revision received March 5, 1971. This work was supported by NASA Goddard under Contract NAS 5-21007. The authors wish to thank J. P. Young under whose direction this work was conducted. They also wish to thank the editors and the anonymous reviewers for their helpful suggestions.

* Senior Staff Analyst.
† Research Staff Engineer.

be considered to be valid. However, the ability to duplicate previously taken test data does not in itself make the model useful.

A useful model should be able to predict the results of untested loading conditions and the effects of changes in the mass, stiffness, or supports of the structure. Such a model would allow the determination of the effects of mission loads, variations in configuration and provide a means for evaluating corrective measures when structural difficulties are anticipated.

By making use of both the quantitative results of testing and the qualitative behavioral characteristics of analysis, it is hoped that such a model may be arrived at. The analysis presented in this paper is considered to be a first step in achieving such a goal.

Description of the Problem

Consider a structure which is to be subjected to dynamic testing. There are certain deflections and rotations of points on the structure that are of interest. It is desired to be able to predict each of these motions under various loading conditions for the structure actually tested and for modified versions of the structure.

It is assumed that an analytical model consisting of P lumped masses interconnected by linear springs with scalar structural damping will be adequate to represent the structure (the more general mass matrix with off-diagonal terms is not excluded). The motion of each of the P masses is considered to represent the motion of P points of the structure including the points of interest. It is also assumed that the loading conditions of interest will contain, primarily, force components at frequencies below some finite value including less than P natural frequencies of the model. It is required that the analytical model faithfully represent the dynamics of the "points of interest" over the "frequency range of interest." It is further required that the model have the capability of predicting the changes in response due to structural changes including mass, stiffness and support changes.

The work reported here is limited in scope as follows. The analysis assumes knowledge of the normal modes at each of P points and the natural frequencies of the structure through the frequency range of interest. It is considered that this data has been obtained from testing. The P points are distributed over the structure so as to represent the characteristics of the measured modes and include all the points of interest. In the rest of the paper, all these points will be referred to as the points of interest.

In the computer experiments described below, the dissipative component of the resonant response was taken as a good approximation to the normal mode. Other more sophisticated procedures such as those given in Refs. 2 or 3 might be justified, for example, if the resonances were not well separated. In addition, it is assumed that a "reasonable" analytical model of the mass matrix of the structure has been derived through analytical or intuitive means. Using these assumed data, a procedure is developed for identifying the parameters in the equations of motion such that the model has the capability of predicting the effects of changes in mass and stiffness on natural frequencies and modes.

Basic Relationships

The matrix equation of the spring-mass-structural damping model just discussed, under steady-state sinusoidal forcing may be written[4]

$$[-\omega^2 M + (1 + ig)K]y = f \tag{1}$$

where y, f are column matrices (vectors) representing the complex amplitudes of displacement and applied force at each of the P points of interest. M and K are $P \times P$ symmetric matrices representing the mass and stiffness coefficients of the model and g is the scalar structural damping coefficient.

The following brief development, while not novel, is carried out here to facilitate the presentation to follow.

The impedance matrix, Z, and the mobility matrix, Y, may be defined from $Z\dot{y} = f$ and $\dot{y} = Yf$ where $\dot{y} = i\omega y$ represents the complex velocity amplitudes. From Eq. (1), these may be written

$$Z = \{(g/\omega)K + i[\omega M - (1/\omega)K]\} \tag{2}$$

and

$$Y = Z^{-1} = \{(g/\omega)K + i[\omega M - (1/\omega)K]\}^{-1} \tag{3}$$

The eigenvalue problem from Eq. (1) is

$$(K - \Omega_i^2 M)\phi_i = 0 \qquad i = 1,2, \ldots P \tag{4}$$

where Ω_i are the natural frequencies and ϕ_i are the normal modes of the system. This may be written

$$M^{-1}K\phi_i = \Omega_i^2\phi_i \qquad i = 1,2, \ldots P \tag{5}$$

and where K is nonsingular ($C = K^{-1}$)

$$CM\phi_i = (1/\Omega_i^2)\phi_i \qquad i = 1,2, \ldots P \tag{6}$$

The orthogonality relationship is $\phi_i^T M \phi_j \neq 0$ when $j \neq i$ and $\phi_i^T M \phi_j = m_i$ when $j = i$. It is interesting to note that the normal modes (eigenvectors) of $M^{-1}K$ and its inverse, CM, are the same and that the respective eigenvalues are reciprocals. The dominant mode of $M^{-1}K$ is the one having the highest frequency and the dominant mode of CM is the one having the lowest frequency. Using the square Φ matrix, where $\Phi = [\phi_1\phi_2 \ldots \phi_P]$, this equation may be written as

$$M^{-1}K\Phi = \Phi\lceil\Omega_i\rfloor \tag{7}$$

$$CM\Phi = \Phi\lceil 1/\Omega_i^2\rfloor \tag{8}$$

and the orthogonalitycondition is $\Phi^T M \Phi = \lceil m_i \rfloor$. From Eqs. (7), (8) and the orthogonality equation, it can be shown that

$$K = M\Phi\lceil\Omega_i^2/m_i\rfloor\Phi^T M = \sum_{i=1}^{P} \frac{\Omega_i^2}{m_i} M\phi_i\phi_i^T M \tag{9}$$

$$C = \Phi\lceil 1/\Omega_i^2 m_i\rfloor\Phi^T = \sum_{i=1}^{P} \frac{1}{\Omega_i^2 m_i} \phi_i\phi_i^T \tag{10}$$

The impedance and mobility may also be expressed as summations of the same type:

$$Z = M\Phi\left[\frac{1}{\omega}(g - i)\lceil\Omega_i^2/m_i\rfloor + i\omega\lceil 1/m_i\rfloor\right]\Phi^T M$$

$$= \frac{1}{\omega}\sum_{i=1}^{P} \frac{\Omega_i^2}{m_i}\{g + i[(\omega/\Omega_i)^2 - 1]\} M\phi_i\phi_i^T M \tag{11}$$

$$Y = \omega\Phi\lceil 1/m_i[g\Omega_i^2 + i(\omega^2 - \Omega_i^2)]\rfloor\Phi^T$$

$$= \omega\sum_{i=1}^{P} \frac{g - i[(\omega/\Omega_i)^2 - 1]}{[(\omega/\Omega_i)^2 - 1]^2 + g^2} \frac{1}{\Omega_i^2 m_i} \phi_i\phi_i^T \tag{12}$$

Incomplete Model

The expressions given above for K, C, Z, Y contain summations of simple products of the individual eigenvectors of the system, $\phi_i\phi_i^T$. Each of these is a $P \times P$ square matrix of rank 1. When P of these matrices are summed as indicated, and since the ϕ_i's are linearly independent, the resulting matrix will be of rank P and thus nonsingular.[5] Summing over fewer than all the modes of the model will result in singular matrices.

Each of these terms represents the contribution of the normal mode in question to the matrix being evaluated. If g, M and all the normal modes of the system were known, the behavior of the system could be predicted by forming the complete equation of motion. The question investigated in this paper is this: If incomplete information is available, i.e., only the first N normal modes, is it possible to derive useful information about the behavior of the system and modifications of it? The analytical model described by the incomplete summations will be called an "incomplete model."

The matrices defining the incomplete model are designated K_{inc}, C_{inc}, Z_{inc}, Y_{inc} and are identical to those given in Eqs. (9–12) except that the summations go from 1 to N, $N < P$.

Certain characteristics of the incomplete model are apparent:

1) Since the terms containing the higher values of Ω_i are not included, the dominant terms of K and Z will be missing and thus K_{inc} and Z_{inc} will not resemble the true K and Z matrices.

2) Conversely, the dominant terms of C and Y are included in C_{inc} and Y_{inc}. These are the matrices which represent the responses due to applied forces, and for the model to have validity it is necessary that they approximate the true values for $\omega < \Omega_N$.

3) The four matrices are of order P (and represent the P points of interest) but are of rank N. Thus, they are all singular and they must all be formed separately and not by inversion.

4) The eigenvalue equation from Eq. (9) can be seen to be

$$M^{-1}K_{\text{inc}}\phi_j = \sum_{i=1}^{N} \frac{\Omega_i^2}{m_i}\phi_i\phi_i^TM\phi_j$$
$$= \Omega_j^2\phi_j \qquad j = 1,2, \ldots N$$
$$= 0 \qquad j > N$$

and similarly for CM. Thus it may be said that the incomplete model contains only the first N modes of the corresponding complete model.

As an illustration of the aforementioned characteristic 2, Fig. 1 shows the driving point acceleration response as a function of frequency as might be obtained from a diagonal element of Y and Y_{inc}.

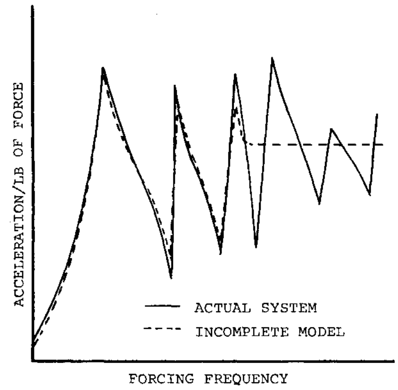

Fig. 1 Typical response of an incomplete model.

Identification of the Mass Matrix

Before proceeding further with the discussion of the incomplete model, a method will be described for identifying the mass matrix. This is an essential step in identifying a useful model of the structure. As above, it is assumed that the first N natural frequencies and normal modes have been determined through testing. Each of these modes contains P elements representing the relative motion of all the points of interest.

The measured normal modes are orthogonal with respect to the unknown mass

$$\phi_i^TM\phi_j = 0 \qquad j \neq i \tag{13}$$

The mass matrix is assumed symmetrical but not necessarily diagonal. Equation (13) is, in reality, $N(N - 1)/2$ independent linear equations having the mass elements as unknowns with products of the elements of the known normal modes as coefficients.

It is possible that certain mass elements may be known or assumed to be zero or to have some definite value. If they are to be zero the corresponding terms are dropped from the equation. If they are to be restricted to a particular value, the corresponding terms are placed on the right-hand side of the equation. If any of the generalized modal masses, m_i, are known, Eq. (13) for $j = i$ can be used and set equal to m_i. Also, the total of the diagonal elements may be considered to be known (the total mass of the structure, for example). This leads to an additional linear equation.

These equations may be written

$$A\bar{m} = R \tag{14}$$

where $\bar{m}$ is a column matrix made up of the unknown elements of M, A is a matrix formed by the coefficients of these unknowns from Eq. (13) and optionally from the other conditions just mentioned. R is a column matrix made up of the right-hand side terms corresponding to known masses, if any, known generalized masses, if any, and possibly the known total mass.

There are, then, at least $N(N - 1)/2$ equations and possibly as many as $N(N + 1)/2 + 1$ if all the equations are used. If none of the mass elements are considered to be known (other than zero), there are at least P unknown diagonal masses and as many as $P(P + 1)/2$ if the matrix is taken to be completely filled. When the number of equations is less than the number of unknowns, there are an infinite number of solutions. When the reverse is true, there will ordinarily be no solution. This treatment will be limited to the first situation where there are an infinite number of solutions to the equations.

In Eq. (14), A is an $n_e \times n_v$ matrix where $n_e < n_v$. $\bar{m}$ is $n_v \times 1$ and R is $n_e \times 1$. The equations have an infinite number of solutions, that is, there are an infinite number of mass distributions that will cause the modes to be orthogonal. In fact, as long as $n_e < n_v$ there will be an infinite number of mass distributions which also give the same generalized masses, m_i, and thus result in the same C_{inc} and Y_{inc} [see Eqs. (10 and 12)]. In other words, different valid mass distributions used with the measured normal modes can predict effectively identical responses of the system to sinusoidal forcing.

If it is desired to use the model for making predictions under other conditions, especially to predict the effects of changes in parameters, it is reasonable to assume that the masses used in the model should be as near to the "true" values as possible. The best information available as to what the "true" values are, is the approximation arrived at by the analyst. These analytical values will not, in general, satisfy the orthogonality condition of the normal modes, i.e., Eq. (14).

The pseudo-inverse[6] is an elegant mathematical tool which will be used to obtain the solution to Eq. (14) which is the closest (in a least-squares sense) to any specified analytical approximation. Another way of thinking of this is that the smallest possible changes in the approximation will be found which are required to satisfy the specified conditions.

Define a column matrix, m_A, which is an approximation to $\bar{m}$ and subtract Am_A from both sides of Eq. (14) giving

$$A(\bar{m} - m_A) = R - Am_A$$

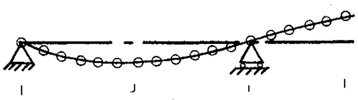

Fig. 2 Beam used for simulated tests.

At this point, a weighting function is introduced in the form of a diagonal matrix, W. Each element is a measure of the analyst's confidence in the corresponding approximation. The result will be that masses having higher values of weighting functions will tend to vary least. Inserting the identity $W^{-1}W$ into the above equation results in

$$(AW^{-1})[W(\bar{m} - m_A)] = R - Am_A$$

Defining $(AW^{-1})^+$ as the pseudoinverse of AW^{-1}, the solution given by

$$W(\bar{m} - m_A) = (AW^{-1})^+(R - Am_A) \quad (15)$$

is the one of the infinite number possible having the smallest weighted sum of squares of the differences of $\bar{m}$ and m_A. This is due to the characteristic property of the pseudoinverse. It may be shown that the pseudoinverse of AW^{-1}, when the rows are linearly independent, is given by

$$(AW^{-1})^+ = W^{-1}A^T[A(W^{-1})^2A^T]^{-1} \quad (16)$$

Then solving (15) for $\bar{m}$,

$$\bar{m} = B_R R + B_m m_A \quad (17)$$

where $B_R = W^{-1}(AW^{-1})^+$ and $B_m = I - B_R A$.

Mass Changes

One of the criteria for evaluating the usefulness of an analytical model is its ability to predict the effects of change. In this section, the use of an incomplete model to predict the effects of mass changes on the natural frequencies and normal modes is discussed.

The stiffness matrix and the influence coefficient matrix are independent of the mass of the system. The expressions derived in terms of the normal modes of the system do contain the mass, however. [See Eqs. (9 and 10).] Thus, it must be concluded that when the mass is changed, the normal modes and frequencies must change in such a way that the summations do not change.

While the expressions for K and C must be invariant when summed over all the modes, they will not be exactly invariant for incomplete summations, i.e., for the incomplete model. Of the two (K and C), it is to be expected that C_{inc} will be less sensitive to mass changes. The reason is that the dominant terms are included in C_{inc} and omitted in K_{inc}, thus C_{inc} is much closer to the invariant matrix C than K_{inc} is to the invariant K.

This hypothesis has been tested by calculating the frequencies and modes of a modified system using matrix iteration on C_{inc} $(M + \Delta M)$. The changes predicted were in excellent agreement with the true values. These results are shown below.

Stiffness Changes

The effect of a change in the stiffness matrix cannot be handled as directly as the mass change discussed in the previous section. Both the K and C matrices must change when the stiffness is changed. Since the dominant terms of K are missing in K_{inc}, it does not appear to be reasonable to hypothesize that $(K + \Delta K)_{\text{inc}} = K_{\text{inc}} + \Delta K$ since even small ΔK's can easily be greater by orders of magnitude than the elements of K_{inc}.

At this point it is necessary to apply the concept of the principal idempotents[7] of a matrix to $M^{-1}K$. Let ϕ_i be a set of vectors, orthogonal with respect to M. Since the ϕ_i's are independent and a_{ij} has as many values as in $M^{-1}K$, it is apparent that $M^{-1}K$ may be written:

Table 1 True mass distribution and approximations

Sta., in.	True mass	Approximations I	Approximations II
0.[a]	0.05		
10.	0.10		
20.	0.10	0.30	0.25
30.	0.10		
40.	0.10	0.15	0.13
50.	0.10	0.10	0.12
60.	0.10	0.10	0.10
70.	0.10	0.10	0.12
80.	0.10	0.15	0.13
90.	0.10		
100.	0.10	0.30	0.25
110.	0.10		
120.[a]	0.10		
130.	0.10		
140.	0.10	0.25	0.15
150.	0.10	0.10	0.13
160.	0.10	0.10	0.12
170.	0.05	0.05	0.05

[a] Support locations.

$$M^{-1}K = \sum_{i=1}^{P} \sum_{j=1}^{P} a_{ij}\phi_i\phi_j^T M \quad (18)$$

To evaluate the a_{ij}'s, premultiply by $\phi_k^T M$ and postmultiply by ϕ_n resulting in

$$\phi_k^T M(M^{-1}K)\phi_n = a_{kn}(\phi_k^T M\phi_k)(\phi_n^T M\phi_n) \equiv a_{kn}m_k m_n$$

or

$$a_{ij} = \phi_i^T M(M^{-1}K)\phi_j/m_i m_j \quad (19)$$

Now, if ϕ_j is an eigenvector of $M^{-1}K$, the numerator of Eq. (19) becomes zero except for $i = j$ and the expression for K becomes identical with Eq. (9). It is important to observe that if the ϕ_i's were not eigenvectors of $M^{-1}K$, there would be coupling terms of the form $a_{ij}M\phi_i\phi_j^T M$ in the expression for K. Thus, if $K + \Delta K$ is expressed in terms of the eigenvectors of $M^{-1}K$, the equation must be of the form

$$K + \Delta K = \sum_{i=1}^{P} \sum_{j=1}^{P} a_{ij}M\phi_i\phi_j^T M \quad (20)$$

Now pre- and postmultiply this equation by ϕ_k^T, ϕ_n and as shown previously it can be seen that

$$\begin{aligned} a_{ij} &= (\Omega_j^2\phi_i^T M\phi_j + \phi_i^T\Delta K\phi_j)/m_i m_j \\ &= \Omega_i^2/m_i + \phi_i^T\Delta K\phi_i/m_i^2, \quad j = i \\ &= \phi_i^T\Delta K\phi_j/m_i m_j, \quad j \neq i \end{aligned}$$

Table 2 Identified masses[a]

Sta., in.	Approx. I m_A	Approx. I $\bar{m}$	Approx. II m_A	Approx. II $\bar{m}$	Approx. II $\bar{m}$ (with error)
20	0.30	0.29	0.25	0.250	0.251–0.257
40	0.15	0.15	0.13	0.134	0.138–0.142
50	0.10	0.11	0.12	0.125	0.125–0.130
60	0.10	0.12	0.10	0.106	0.100–0.109
70	0.10	0.12	0.12	0.124	0.114–0.125
80	0.15	0.15	0.13	0.131	0.122–0.130
100	0.30	0.29	0.25	0.248	0.243–0.249
140	0.25	0.23	0.15	0.146	0.143–0.150
150	0.10	0.08	0.13	0.124	0.120–0.129
160	0.10	0.09	0.12	0.115	0.111–0.117
170	0.05	0.08	0.05	0.047	0.043–0.052
Variance (rms)		0.016		0.0040	0.0044–0.0070

[a] $N = 11$, $P = 3$.

Thus, substituting into Eq. (20), the expression for $K + \Delta K$ can be written

$$K + \Delta K = \sum_{i=1}^{P} \frac{\Omega_i^2}{m_i} M\phi_i\phi_i^T M + \sum_{i=1}^{P} \sum_{j=1}^{P} \frac{\phi_i^T \Delta K \phi_j}{m_i m_j} M\phi_i\phi_j^T M \quad (21)$$

This expression when summed over all P modes is exact. Note that the first summation is equal to K. Now, truncating the series at the last known mode, the expression can be written

$$(K + \Delta K)_{\text{inc}} = K_{\text{inc}} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\phi_i^T \Delta K \phi_j}{m_i m_j} M\phi_i\phi_j^T M \quad (22)$$

This expression can now be evaluated and it is hypothesized that $M^{-1}(K + \Delta K)_{\text{inc}}$ can be used to obtain good approximations to the new natural frequencies and normal modes. This hypothesis has been tested and the results given below appear quite satisfactory.

Testing the Theory

The procedure suggested for identifying the parameters of a valid and useful model are based on a combination of mathematical rigor and reasonable hypothesis. The basic equations are rigorous. The capability of the incomplete model to predict the effects of mass and stiffness changes and the suggested procedures for carrying out these computations are hypothesized to be adequate. These hypotheses are based on what are considered to be reasonable arguments. It is obviously necessary to test the theory if one is to have confidence in the methods proposed.

A computer program was written to generate simulated test data (including error), identify the mass matrix, and form K_{inc} and C_{inc}. The program then accepts mass changes and stiffness changes and computes the changes in eigenvalues and eigenvectors as described in the previous section.

The specimen selected for testing was a simple, thin overhung beam in transverse bending as shown in Fig. 2. The beam had a uniform EI and 18 lumped masses approximating a uniform mass distribution and a damping coefficient of 0.02. This specimen was selected as being not so simple that success would give no confidence in the methods and yet not so complex that one would not have a good intuitive understanding of the characteristics of the system. The number of points of interest were taken to be 11 and the number of modes as 3.

The simulated testing was carried out as follows: 1) the 18×18 K matrix for the beam was calculated and the exact natural frequencies were computed; 2) the system was considered to be driven sinusoidally at station 60 at the first three natural frequencies; 3) the real velocity response at the eleven points of interest were computed; 4) in certain runs, a measurement error was introduced in the form of a bias plus a randomly distributed error; 5) these simulated measurements were considered to be measured normal modes. The parameter identification was then performed as follows: 1) two different but reasonable assumed mass distributions (see next section) were used and the procedure previously discussed was applied to obtain the mass matrix; and 2) the K_{inc} and C_{inc} matrices were calculated. The ability to predict the effects of changes was then tested by: 1) assuming that a large lumped mass was added at the tip and 2) assuming that a spring was added to ground at the tip. The computed changes in natural frequencies and normal modes were compared with the exact values for these changes. Further details are given in the following sections.

Table 3 Predicted effects of tip mass on frequency

Mode	1	2	3
Frequency tested	8.32	18.65	49.06
Exact new frequency	3.82	14.77	47.06
Approx. I	4.08	14.87	47.48
Approx. II	3.99	14.84	47.37
II with error	3.83–4.05	14.69–15.00	47.29–47.46

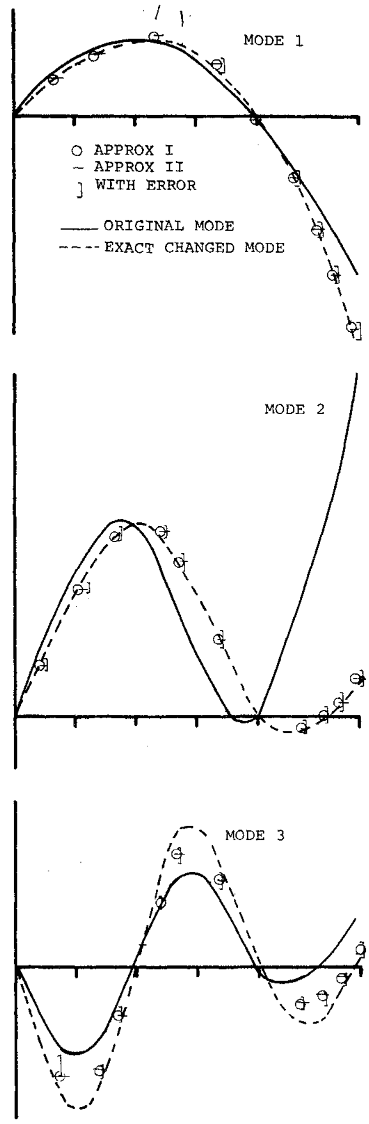

Fig. 3 Predicted changed modes due to added mass at tip of beam.

Identified Masses

Two different, but reasonable, mass approximations were used as shown in Table 1. The differences are that in approximation I, the masses at the supports are lumped at the adjacent points of interest, while in approximation II these masses are ignored. In approximation II the masses are distributed over several points rather than being lumped at the nearest point of interest.

For the cases shown, the following conditions were imposed: 1) only diagonal masses were assumed; 2) the total mass was constrained to be constant; and 3) the weighting function was kept equal to unity throughout. For approximation I, no error was assumed. Approximation II was run without error and also with error. The case with error assumed +5% bias plus a randomly distributed error between −5% and +5% of the amplitude (equivalent to a random error between 0 and +10%). The case with error was run five times under the same nominal conditions but

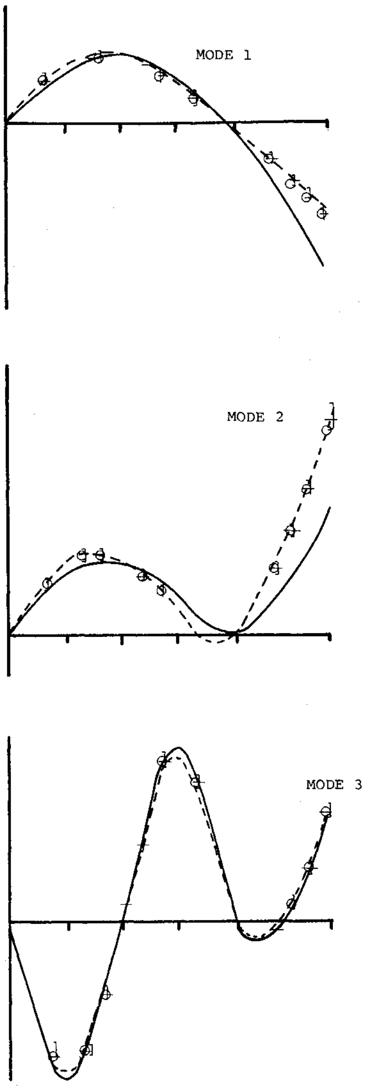

Fig. 4 Predicted changed modes due to added spring at tip of beam.

with different random sequences. This data is shown as a spread over the range of values obtained.

These results are illustrated in Table 2. Also given are the variances between m_A and $\bar{m}$ in rms. This is the quantity that is minimized in the procedure. Aside from the fact that all the distributions computed are reasonable, there is one characteristic which justifies comment. Approximation II is intuitively considered to be a "better" representation than approximation I. Notice (by observing the variances) that the "better" approximation required smaller changes to satisfy the orthogonality condition. If an inordinately large variance were found in a practical application, one would be led to the conclusion that the intuitive model was inadequate; for example, a significant coordinate or mass item had been omitted.

While the identified masses all appear reasonable and relatively insensitive to measurement error, the proof of the usefulness of the model will come when the effects of changes are evaluated in the next sections.

One further comment which might be of interest: It is tempting to think, at first glance, that one might improve the mass distribution further by an iterative process using the computed $\bar{m}$ as an approximation and recomputing a new $\bar{m}$. This, of course, cannot work since any $\bar{m}$ obtained satisfies the orthogonality conditions and no further change is required. Thus, this process might be thought of as an iteration process which converges absolutely in one iteration and where the converged solution depends on the first trial used.

Table 4 Predicted effects of tip spring on frequency

Mode	1	2	3
Frequency tested	8.32	18.65	49.06
Exact new frequency	11.25	21.23	49.20
Approx. I	11.00	20.84	49.19
Approx. II	11.11	20.99	49.20
II with error	11.02–11.38	20.80–21.25	49.20–49.22

It can be readily shown that [see Eq. (12)] if $m_1 = B_R R + B_m m_A$, then $\bar{m}_2 = B_R R + B_m \bar{m}_1 = \bar{m}_1$.

Effect of a Mass Change

A mass was considered added at the free tip of the beam. The mass was 1.0 lb-sec²/in. which was an addition of approximately 60% of the original mass of the beam. The frequency changes were quite large.

Table 3 shows the original natural frequencies of the beam (the "frequencies tested"), the exact new frequencies, and the frequencies predicted by the method described above. All the predictions are quite adequate including those which contained measurement error. Notice that the results for the "good" mass approximation are slightly better than those obtained using approximation I.

The predicted normal modes compared to the original and the exact new ones are shown in Fig. 3. In each case, they are normalized at station 60, the driving point. Notice that the new modes 1 and 2 are quite adequately predicted while the third mode is not predicted as well. It may be presumed that the new third mode will contain components of the old fourth mode which has not been included in this analysis. It appears that the predicted change in the highest frequency may be better than the change in the corresponding mode shape.

Effect of a Stiffness Change

A stiffness change consisting of the addition of a 1000 lb/in. spring from the tip to ground was considered. This was handled by computing ΔK_{inc} as discussed in the text.

Table 4 shows the results of the frequency calculation. The same conclusions as those regarding the mass change effects are appropriate.

Figure 4 shows the predicted and exact normal modes. All the predictions appear quite adequate including that of the highest mode. This is probably due to the fact that this particular mode changed very little.

Conclusions

This paper presented the concept of an incomplete model based on fewer measured normal modes than spatial coordinates. The matrices resulting are singular and thus must be handled with caution. It has been shown that it is possible to combine the quantitative information from tests and the qualitative information from analysis to form a valid and useful model.

The usefulness has been demonstrated by computer experiments. Large changes in natural frequencies and normal modes due to mass and constraint changes were quite satisfactorily predicted even when the data was polluted with simulated test errors. It appears that further applications and expansion in the scope of the concepts presented are justified.

References

[1] Young, J. P. and On, F. J., "Mathematical Modeling Via Direct Use of Vibration Data," Preprint 690615, Oct. 1969, National Aeronautic and Space Engineering and Manufacturing Meeting, Society of Automotive Engineers, Los Angeles, Calif.

[2] Flannelly, W. G., Berman, A., and Barnsby, R. M., "Theory of Structural Dynamic Testing Using Impedance Techniques," USAAVLABS TR 70-6A,B, June 1970, U. S. Army Aviation Materiel Labs., Fort Eustis, Va.

[3] Kennedy, C. C. and Pancu, C. D. P., "Use of Vectors in Vibration Measurement and Analysis," *Journal of the Aeronautical Sciences*, Vol. 14, No. 11, Nov. 1947, pp. 603–625.

[4] Scanlan, R. H. and Rosenbaum, R., *Introduction to the Study of Aircraft Vibration and Flutter*, Macmillan, New York, 1951.

[5] Bodewig, E., *Matrix Calculus*, Interscience, New York, 1956.

[6] Greville, T. N. E., "The Pseudoinverse of a Rectangular or Singular Matrix and its Application to the Solution of Systems of Linear Equations," *SIAM Review*, Vol. 1, No. 1, Jan. 1959.

[7] Rosanoff, R. A., "A Survey of Modern Nonsense as Applied to Matrix Computations," *Volume of Technical Papers on Structural Dynamics, AIAA Structural Dynamics and Aeroelasticity Specialist Conference*, AIAA, New Orleans, 1969, pp. 275–284.

AUGUST 1971 AIAA JOURNAL VOL. 9, NO. 8

Optimization of Complex Structures to Satisfy Flutter Requirements

CARL S. RUDISILL* AND KUMAR G. BHATIA†
Clemson University, Clemson, S. C.

Equations for finding the partial derivatives of the flutter velocity of an aircraft structure with respect to structural parameters are derived. A numerical procedure is developed for determining the values of the structural parameters such that a specified flutter velocity constraint is satisfied and the structural mass is a relative minimum. A search procedure is presented which utilizes two gradient search methods and a gradient projection method. The procedure is applied to the design of a box beam.

I. Introduction

STRUCTURAL design is often accomplished through a series of design iterations, in which a trial design is chosen, analyzed and modified by the designer after examination of the numerical results. This iterative process does not guarantee a design of minimum weight for all design conditions since the designer's judgment and intuition are influencing factors in the redesign process and only a small number of design iterations are practical. Even with the aid of the digital computer, the design of complex structures to satisfy dynamic response restrictions has been hampered by the inherent difficulty and the computational cost of dynamic analysis. Recent advances in programing techniques, and matrix methods of structural analysis have provided all the necessary tools for the development of efficient structural optimization methods.

During the last few years numerous publications have appeared in the literature[1–4] dealing with the dynamic optimization of structures. M. J. Turner[1] developed a procedure for determining the relative proportions of selected elements of an aircraft structure to attain a specified flutter speed with minimum total mass. Lagrange multipliers were employed to introduce the conditions for neutral dynamic stability (flutter equations) as dynamic constraints. The resulting system of nonlinear equations was solved by the Newton-Raphson process to determine the masses of the elements of the system.

McCart, Haug and Streeter[2] developed a steepest-descent boundary-value method for the design of structures with constraints on strength and natural frequency. A computational algorithm was developed which implemented the steepest-descent method. The method was developed in detail for a three-member frame design problem and references were given for a more general development.

Fox and Kapoor[3] developed a structural optimization procedure in which limitations were imposed upon maximum dynamic stresses and displacements (handled by the shock spectral approach) as well as the natural frequencies of the structure. A direct optimization method (the method of feasible directions) which consisted of a design-analysis cycle was used. Exact and computationally efficient schemes were developed for finding derivatives of maximum response and natural frequencies.

C. P. Rubin[4] developed a procedure for the determination of the least weight structure for a specified natural frequency requirement. The method modified an existing structure by varying the cross-sectional properties of its members. This was accomplished by using gradient equations to first obtain the desired structural frequency, and then separate gradient equations were used to decrease the weight of the structure while the natural frequency was held constant.

Most of the important papers dealing with the subject of optimum design of structures for dynamic requirements are listed as references in the works cited. The methods used in Refs.[1–4] appear to be most promising and improvement and extension of these methods to include aeroelastic constraints such as divergence speed, reversal of control speed and flutter speed appears feasible. In principle there are many ways in which an optimum design of a structure may be found; essentially the problem is that of developing an efficient design procedure which requires a reasonable amount of computer storage and run time to find the optimum design of a complex structure.

II. Description of Optimization Procedure

The primary objective of this paper is to develop a numerical procedure for determining a selected number of structural dimensions or parameters for the optimum (minimum mass) design of a complex structure with a specified flutter velocity. Structural parameters (cross-sectional areas, plate

Received December 21, 1970. The authors wish to express their appreciation to R. V. Doggett, Jr. of the NASA Langley Research Center for his many valuable suggestions during the course of this work. This research was supported by NASA Research Grant NGR-41-001-027.

* Associate Professor of Mechanical Engineering. Member AIAA.

† Graduate Student. Student Member AIAA.